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A Bayesian significance test of the stationarity of regression parameters

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SUMMARY

This paper presents a Bayesian significance test for stationarity of a regression equation using the highest posterior density credible set. In addition, a solution to the Behrens–Fisher problem is provided. From a Monte Carlo simulation study, it has been shown that the Bayesian significance test has stronger power than the Cusum and the Cusum of squares tests. The Bayesian significance test may be useful in detecting individual parameter nonstationarity.

Some key words: Behrens–Fisher problem; Change-point; Highest posterior density credible set; *p*-value.

1. INTRODUCTION

Consider the univariate switching regression models

$$Y_1 = X_1\beta_1 + u_1, \quad Y_2 = X_2\beta_2 + u_2. \qquad (1\cdot 1)$$

Here,

$$Y_1 = (y_1, \ldots, y_\tau), \quad Y_2 = (y_{\tau+1}, \ldots, y_T),$$

where τ is the unknown change-point. The X_i $(i=1,2)$ is an $n_i \times p$ design matrix, where $n_1 = \tau$ and $n_2 = T - \tau$. The u_i $(i=1,2)$ is n_i-variate normally distributed with zero mean vector and covariance matrix $\sigma_i^2 I_{n_i}$, and $\beta_i \in R^p$.

Testing for stationarity in the above regression models involves the null hypothesis $\beta_1 = \beta_2$ and $\sigma_1^2 = \sigma_2^2$ against the alternative that $\beta_1 \neq \beta_2$ or $\sigma_1^2 \neq \sigma_2^2$ with the unknown change-point τ. If τ is known, this is the Behrens–Fisher problem, testing equality of regression coefficient vectors in two regression equations with unequal error variances.

Among classical approaches, Quandt (1958, 1960) used maximum likelihood to estimate τ and applied the likelihood ratio test conditionally on the estimated τ. Brown, Durbin & Evans (1975) suggested the Cusum and the Cusum of squares tests using recursive residuals. Tsurumi (1977), Choy & Broemeling (1980) and Holbert (1982) provided a traditional Bayes testing procedure using posterior odds, assigning a point mass prior probability to the null hypothesis $\tau = T$ and a uniform prior probability over $\{1, \ldots, T-1\}$.

It is well known, however, that the traditional Bayes test using the posterior odds is sensitive to the prior inputs in most cases. The Bayes test presumes that the null value of a hypothesized parameter is believed to be stronger than any other value near the null value, a presumption which is sometimes true, but in many applications unsupported. A local sensitivity analysis shows in Appendix 1 that the posterior features are very sensitive to the prior inputs in testing the stationarity of the regression parameter set in $(1\cdot 1)$. So, robustness is of concern.

For these circumstances, Lindley (1965, p. 61) and Box & Tiao (1965; 1973, p. 80) suggested a Bayesian significance test through the construction of the highest posterior density credible set, and emphasized that this type of significance test is appropriate only for circumstances in which prior knowledge of the hypothesized parameter is vague or diffuse, and one of the hypotheses is a single point. Within this framework, Hsu (1982) calculated the posterior probability of parameter differences between two regimes conditionally on the estimated change-point in the exponential power distribution model.

In many cases, we have no prior information on stationarity of regression parameters. Within the highest posterior density credible set framework, this paper develops a Bayesian significance test for stationarity of regression parameters over a sequence of time series by deriving the posterior distribution of $\beta_1-\beta_2$ and σ_2^2/σ_1^2 unconditionally on the change-point τ. This test does not require a constant error variance assumption, and does not assume a change-point fixed at the estimated point.

This paper is organized as follows. In § 2, the Bayesian significance tests for stationarity of a regression equation and an individual coefficient are suggested. An application to testing regression equality under unequal error variances or the Behrens–Fisher problem is discussed in § 3. In order to examine the testing power of the Bayesian significance test, the Cusum test, and the Cusum of squares test, and the Quandt likelihood ratio test are considered as competitors. A simulation is presented for the power comparisons in § 4. An example in § 5 illustrates how one applies the Bayesian significance test in practice.

2. A BAYESIAN SIGNIFICANCE TEST

The null hypothesis H_0 that there is no structural shift of the regression equation in (1·1) over a sample period with T observations taken sequentially is, without reference to τ,

$$\delta=\beta_1-\beta_2=0, \quad \rho=\sigma_2^2/\sigma_1^2=1.$$

For the Bayesian significance test, therefore, the posterior distributions of δ and ρ are needed to obtain the confidence region, i.e. highest posterior density credible set, of δ and ρ.

One has a parameter set $\Theta=(\tau,\beta_1,\beta_2,r_1,r_2)$, where $r_i=1/\sigma_i^2$. Since prior knowledge of $\Theta'=(\beta_1,\beta_2,r_1,r_2)$ is often vague or diffuse, we employ a diffuse prior for Θ', Θ' and τ being assumed independent. The prior distribution of Θ is, therefore,

$$\pi(\Theta)\propto\pi(\tau)\frac{1}{r_1r_2},$$

where $\pi(\tau)$ is the prior distribution of τ. Note that the functional forms $\pi(.)$ and $\pi(.|.)$ represent a prior and a posterior distribution, respectively. The posterior distribution of the change-point τ is obtained by integrating out $\pi(\Theta\,|\,Y,X)$ with respect to Θ':

$$\pi(\tau\,|\,Y,X)\propto\pi(\tau)\prod_{i=1}^{2}\Gamma(\tfrac{1}{2}n_i-\tfrac{1}{2}p)|X_i'X_i|^{-\frac{1}{2}}(\text{SSE}_i)^{-\frac{1}{2}(n_i-p)}, \tag{2·1}$$

where $\text{SSE}_i=(Y_i-X_i\hat{\beta}_i)'(Y_i-X_i\hat{\beta}_i)$, and $\hat{\beta}_i$ is the ordinary least squares estimator of β_i in the ith regime, for $i=1, 2$.

The conditional posterior distributions of β_i and ρ_i given τ can also be obtained by integrating out $\pi(\Theta \mid Y, X)$.

We now state the following theorem. For a proof see Appendix 2.

THEOREM. (*i*) *Given τ and ρ, the conditional posterior distribution of δ is*

$$\pi(\delta \mid \tau, \rho, y, x) \propto \left\{1 + (\delta - \hat{\delta})' \frac{M}{(T-2p)s^2}(\delta - \hat{\delta})\right\}^{-\frac{1}{2}(T-p)}, \tag{2·2}$$

where

$$M = X_2'X_2(\rho X_1'X_1 + X_2'X_2)^{-1}\rho X_1'X_1, \quad s^2 = \frac{\rho \text{SSE}_1 + \text{SSE}_2}{T-2p},$$

which is the p-variate Student t distribution with location vector $\hat{\delta} = \hat{\beta}_1 - \hat{\beta}_2$, precision matrix M/s^2, and $(T-2p)$ degrees of freedom. Equivalently, the quantity

$$F(\delta) = \frac{Q(\delta)}{ps^2} = \frac{(\delta - \hat{\delta})'M(\delta - \hat{\delta})}{ps^2} \tag{2·3}$$

is distributed a posteriori as a conditional F-variable with $(p, T-2p)$ degrees of freedom given τ and ρ.

(*ii*) *Given τ, the conditional posterior distribution of $\rho' = (\hat{\sigma}_1^2/\hat{\sigma}_2^2)\rho$ is an F distribution with $(n_1 - p, n_2 - p)$ degrees of freedom, where*

$$\hat{\sigma}_1^2 = \text{SSE}_1/(n_1 - p), \quad \hat{\sigma}_2^2 = \text{SSE}_2/(n_2 - p).$$

The unconditional posterior distributions of an $F(p, T-2p)$ random variable $F(\delta)$ and ρ are, respectively,

$$\begin{aligned} \pi(F(\delta) \mid Y, X) &= \sum \left\{\int_0^\infty \pi(F(\delta) \mid \tau, \rho, Y, X)\pi(\rho \mid \tau, Y, X)\, d\rho\right\}\pi(\tau \mid Y, X), \\ \pi(\rho \mid Y, X) &= \sum \pi(\rho \mid \tau, Y, X)\pi(\tau \mid Y, X). \end{aligned} \tag{2·4}$$

The null hypothesis H_0 can be divided into two sub-nulls $H_{01}: \delta = 0$ and $H_{02}: \rho = 1$, and H_0 could be rejected if either of these two sub-nulls is rejected. The separation of the null into two sub-nulls would be helpful to determine which parameters are nonstationary. One defines separately the highest posterior density credible sets of $F(\delta)$ and ρ based on conditional distributions since $F(\delta)$ and ρ are conditionally independent. But these credible sets will be used only to define the unconditional p-value and thereby an unconditional test.

Given ρ and τ, the $(1-\alpha)$ credible set for $F(\delta)$ is defined as

$$C_{F(\delta)} = \{F(\delta): 0 < F(\delta) < F(p, T-2p; 1-\alpha)\},$$

where $F(p, T-2p; 1-\alpha)$ is the $(1-\alpha)$th quantile of an F-distribution with $(p, T-2p)$ degrees of freedom. Hence, given ρ and τ, the decision rule for H_{01} is to reject if $F(0) = (ps^2)^{-1}\hat{\delta}'M\hat{\delta} \in \bar{C}_{F(\delta)}$, where $\bar{C}_{F(\delta)}$ is the complement of $C_{F(\delta)}$. The unconditional p-value of H_{01}, therefore, is calculated from (2·4) to yield

$$\begin{aligned} p_{\delta=0} &= 1 - E_\tau(E_\rho[\mathscr{F}_{p,T-2p}\{F(0)\}]) \\ &= \sum_\tau \left(\int_0^\infty [1 - \mathscr{F}_{p,T-2p}\{F(0)\}]\pi(\rho \mid \tau, Y, X)\, d\rho\right)\pi(\tau \mid Y, X), \end{aligned} \tag{2·5}$$

where $\mathscr{F}_{p,T-2p}$ is the cumulative density function of an F distribution with $(p, T-2p)$ degrees of freedom, and the expectations E_τ and E_ρ are taken with respect to τ and ρ, respectively. The integral with respect to ρ is evaluated numerically. The p-value defined in (2·5) is the weighted average of conditional p-values given τ. Our test, therefore, is to reject H_{01} if $p_{\delta=0}$ falls below α. This test results in a size α test.

Likewise, the unconditional p-value of H_{02} is

$$p_{\rho=1} = 2\sum_\tau [1-\mathscr{F}_{n_1-p,n_2-p}\{\max(\rho_0', 1/\rho_0')\}]\pi(\tau \mid Y, X), \tag{2·6}$$

where $\rho_0' = \hat{\sigma}_1^2/\hat{\sigma}_2^2$.

It is often necessary to identify an individual nonstationary element of the coefficient vector. In order to test the null hypothesis H_0^k that the kth regression coefficient is stationary, namely, H_0^k: $\delta_k = \beta_{1k} - \beta_{2k} = 0$ unconditionally on τ, we need the posterior distribution of δ_k. From (2·2), the conditional posterior distribution of δ_k on τ and ρ is

$$\pi(\delta_k \mid \tau, \rho, Y, X) \propto \left\{1 + \frac{c_k(\delta_k - \hat{\delta}_k)^2}{(T-2p)s^2}\right\}^{-\frac{1}{2}(T-2p+1)},$$

where c_k is the reciprocal of the kth diagonal element of M^{-1}, and $\hat{\delta}_k = \hat{\beta}_{1k} - \hat{\beta}_{2k}$. The unconditional p-value for H_0^k is defined as

$$\begin{aligned} p_{\delta_k=0} &= 2(1 - E_\tau[E_\rho\{\mathscr{T}_{T-2p}|t(0)|\}]) \\ &= 2\sum_\tau \left[\int_0^\infty \{1-\mathscr{T}_{\tau-2p}|t(0)|\}\pi(\rho \mid \tau, Y, X)\, d\rho\right]\pi(\tau \mid Y, X), \end{aligned} \tag{2·7}$$

where $\mathscr{T}_{T-2p}$ is the cumulative density function of the standard Student t distribution with $(T-2p)$ degrees of freedom, $t(0) = -c_k^{\frac{1}{2}}\hat{\delta}_k/s$. Note that $t(\delta_k) = c_k^{\frac{1}{2}}(\delta_k - \hat{\delta}_k)/s$ is the standard Student t variable.

In computing δ_k given τ, $\pi(\delta_k \mid \tau, Y, X)$, Patil's approximation (Patil, 1964, 1965; Box & Tiao, 1973, p. 105) may be employed in the univariate case only, in which δ_k is approximated as a Student t variable conditionally on τ, where both β_{1k} and β_{2k} are independent Student t variables. However, Patil's approximation is not appropriate in testing the multivariate case like in $H_{01}: \beta = 0$, and it is of little accuracy when the degrees of freedom are small. Moreover, for the existence of the precision parameter of δ_k, the sample size of each regime should be at least 7 in the simple regression models.

3. Application to testing equality with unequal variances

As mentioned earlier, when the change-point is known, testing for stationarity of a regression equation becomes the Behrens–Fisher problem or testing problem for equality of regression coefficient vectors in two regression equations with unequal variances.

Chow (1960) proposed a test for equality of coefficients in two regression equations, namely, $\delta = 0$, when error variances are equal ($\rho = 1$). In this case, the test quantity $F(0)$ from (2·3) in the Bayesian significance test is the same as the Chow test statistic. The Chow test is an exact test under homoscedasticity ($\rho = 1$), but is not under heteroscedasticity. After Chow, many researchers attempted to solve the equality testing problem under heteroscedasticity. Most require the knowledge of the ratio ρ of error variances or are conditioned on an estimate of ρ. The Bayesian significance test, however, does not require knowledge of ρ.

The test for the sub-null $H_{01}: \delta = 0$ can be applied to the Behrens–Fisher problem. With fixed τ as the sample size of the first population, a Bayesian solution for the p-variate Behrens–Fisher problem could be obtained from (2·5) with desired accuracy. Note that $\pi(\tau)$ is degenerate at the fixed τ with probability 1. The p-value of the Bayesian solution is

$$p_{\delta=0} = \int_0^\infty [1 - \mathscr{F}_{p,T-2p}\{F(0)\}]\pi(\rho \mid Y, X)\, d\rho. \tag{3·1}$$

Weerahandi (1987) also provided a solution from the frequentist viewpoint without assuming knowledge of the ratio of error variances. In fact, Weerahandi's solution parallels numerically the Bayesian solution, since the p-value of Weerahandi's solution (1987, eqn (12)) equals (3·1).

4. Simulation study for the comparison of power

Simulation has been used to compare the power of the Bayesian significance test with those of the Quandt likelihood test (Quandt, 1958, 1960), the Cusum test, and the Cusum of squares test (Brown et al., 1975). Note that the Cusum and the Cusum of squares tests are not conditioned on τ, whereas the Quandt likelihood test is.

We consider the following simple regression settings:

$$y_t = \alpha_1 + \beta_1 x_t + u_t \quad (t = 1, \ldots, \tau),$$
$$y_t = \alpha_2 + \beta_2 x_t + u_t \quad (t = \tau + 1, \ldots, T),$$

where the u_t of the ith regime, for $i = 1, 2$, are independently normally distributed with mean 0 and variance σ_i^2.

To compare the powers, each testing procedure is applied for each simulated data set in which the magnitudes of parameter shift are $(\delta_\alpha, \delta_\beta, \rho)$, and one computes the rejection rate of the null at a significance level. The simulated data sets are formed as follows; the sample size is set to 40 and the change-point τ is randomly sampled from $\{3, \ldots, 37\}$. One considers 7 cases for δ_α, 9 cases for δ_β, and 5 cases for ρ. The total cases per replication is 315 and the number of replications is 1000.

Among the testing procedures considered, the Quandt likelihood ratio test is an asymptotic test. It is worthwhile, therefore, to examine the actual significance levels of the Quandt likelihood ratio test which were 0·376, 0·372, 0·370, 0·365 and 0·362 for sample sizes 20, 30, 40, 50 and 60, respectively, at 5% nominal significance level. At 1% nominal significance level, they are 0·181, 0·178, 0·170, 0·170 and 0·168, respectively. These numbers mean that the Quandt likelihood ratio test rejects the null too frequently, and that the convergence rate is slow as the sample size increases. These facts are enough to drop the Quandt likelihood ratio test.

Table 1 is the summary of the simulation results. Numbers represent the average rejection rate of the null by each testing procedure for the given value of a parameter shift. For example, the average rejection rate at a given significance level when the magnitude of the intercept shift is δ_α is defined as

$$\bar{P}(\delta_\alpha) = \frac{1}{N(\delta_\beta)N(\rho)} \sum_{L=1}^{N(\delta_\beta)} \sum_{L=1}^{N(\rho)} P(\delta_\alpha, \delta_\beta, \rho), \tag{4·1}$$

where $N(\delta_\beta)$ and $N(\rho)$ are the numbers of cases considered of δ_β and ρ, respectively, and $P(\delta_\alpha, \delta_\beta, \rho)$ is the rejection rate when the parameter shifts are given δ_α, δ_β and ρ. The average rejection rates for δ_β and ρ can be defined similarly. The significance level is 5%.

Table 1. *Average rejection rate of H_0 by each testing procedure at 5% significance level*

(*a*) $\delta_\alpha = \alpha_1 - \alpha_2 = 0$

Test	−0·90	−0·60	−0·30	0·00	0·30	0·60	0·90
B	0·956	0·918	0·827	0·740	0·830	0·917	0·955
C	0·242	0·158	0·088	0·063	0·090	0·160	0·224
CS	0·839	0·834	0·826	0·814	0·826	0·835	0·841

(*b*) $\delta_\beta = \beta_1 - \beta_2 = 0$

Test	−1·20	−0·90	−0·60	−0·30	0·00	0·30	0·60	0·90	1·20
B	0·959	0·931	0·885	0·805	0·738	0·803	0·886	0·931	0·959
C	0·198	0·163	0·145	0·123	0·097	0·119	0·141	0·161	0·197
CS	0·903	0·871	0·827	0·773	0·722	0·773	0·830	0·872	0·906

(*c*) $\rho = \sigma_2^2/\sigma_1^2 = 1$

Test	$\frac{1}{10}$	$\frac{1}{5}$	1	5	10
B	0·870	0·883	0·884	0·882	0·869
C	0·149	0·148	0·146	0·149	0·154
CS	0·824	0·833	0·835	0·834	0·828

B, Bayesian significance test; C, Cusum test; CS, Cusum of squares test.
Number of replications for each set of parameter shifts, 1000.

Table 1 shows that the Bayesian significance test has stronger power over most of the range of $(\delta_\alpha, \delta_\beta, \rho)$ than the Cusum and Cusum of squares tests. The Cusum test has the weakest power. The actual significance levels of the Bayesian significance, the Cusum, and the Cusum of squares tests, i.e. the rejection rate of the null when $\delta_\alpha = \delta_\beta = 0$ and $\rho = 1$, are 0·052, 0·044 and 0·051, respectively.

Table 2. *Average rejection rate of each subnull of individual parameter stationarity by the Bayesian significance test at 5% level*

(*a*) $\delta_\alpha = \alpha_1 - \alpha_2 = 0$

Subnull	−0·90	−0·60	−0·30	0·00	0·30	0·60	0·90
H_0^1: $\delta_\alpha = 0$	0·922	0·736	0·414	0·033	0·412	0·744	0·921
H_0^2: $\delta_\beta = 0$	0·597	0·645	0·680	0·694	0·682	0·646	0·601
H_{02}: $\rho = 1$	0·438	0·414	0·385	0·376	0·389	0·411	0·442

(*b*) $\delta_\beta = \beta_1 - \beta_2 = 0$

Subnull	−1·20	−0·90	−0·60	−0·30	0·00	0·30	0·60	0·90	1·20
H_0^1: $\delta_\alpha = 0$	0·659	0·638	0·615	0·561	0·442	0·557	0·616	0·634	0·655
H_0^2: $\delta_\beta = 0$	0·939	0·877	0·720	0·375	0·027	0·374	0·718	0·875	0·939
H_{02}: $\rho = 1$	0·476	0·435	0·394	0·359	0·345	0·355	0·388	0·437	0·481

(*c*) $\rho = \sigma_2^2/\sigma_1^2 = 1$

Subnull	$\frac{1}{10}$	$\frac{1}{5}$	1	5	10
H_0^1: $\delta_\alpha = 0$	0·595	0·591	0·593	0·601	0·607
H_0^2: $\delta_\beta = 0$	0·633	0·657	0·669	0·657	0·631
H_{02}: $\rho = 1$	0·576	0·345	0·203	0·346	0·569

Numbers of replication for each set of parameter shifts, 1000.

Another advantage of the Bayesian significance test over the Cusum and Cusum of squares tests is the possibility of testing the stationarity of an individual parameter. To do this, we test the subnulls

$$H_0^1\colon \delta_\alpha = \alpha_1 - \alpha_2 = 0, \quad H_0^2\colon \delta_\beta = \beta_1 - \beta_2 = 0, \quad H_{02}\colon \rho = \sigma_2^2/\sigma_1^2 = 1$$

with the previous simulated data. Each p-value can be computed through (2·6) and (2·7). On the other hand, the Cusum and the Cusum of squares tests can hardly be applied to stationarity testing of individual parameter.

Table 2 presents the average rejection rate of each subnull by the Bayesian significance test. Table 2 shows how sensitive the Bayesian significance test is to a shift in the target parameter. For example, the Bayesian significance test for the subnull H_0^2 is sensitive to the difference, δ_β, in the slopes, while the rejection rate of the test for subnull H_0^1 or H_{02} does not change much over the range of δ_β considered. Likewise, the Bayesian significance test for the subnull H_0^1 is sensitive to δ_α, while the testing power of the test for the other parameter stationarity is insensitive to the value of δ_α. The same is found in testing the stationarity of the error variances. Thus, the Bayesian significance test for stationarity of an individual parameter is relatively sensitive to a change in the target parameter, but insensitive to a change in the other parameters, and that the Bayesian significance test may be useful to determine which parameter is nonstationary.

5. An illustration of application

An example illustrates how one may employ the Bayesian significance test to detect a nonstationary parameter of Quandt's (1958) changing linear regression model

$$y_t = 2{\cdot}5 + 0{\cdot}7x_t + \varepsilon_t \quad (t = 1, \ldots, 12),$$
$$y_t = 5{\cdot}0 + 0{\cdot}5x_t + \varepsilon_t \quad (t = 13, \ldots, 20).$$

The ε_t's are independent and identical $N(0, 1)$. The model shifts at the 12th observation, and $\delta_\alpha = -2{\cdot}5$, $\delta_\beta = 0{\cdot}2$ and $\rho = 1$. The absolute magnitude of shift in the intercept is relatively large to the shift in the slope. But, the error variances do not change.

Table 3 lists the Quandt data set, the conditional p-values on τ of the Bayesian significance test for each subnull H_0^1, H_0^2 and H_{02}, and then unconditional p-values for these subnulls. Based on the unconditional p-values, stationarity of the intercept and the slope is obviously rejected at 5% and 10% significance levels, respectively. The smaller p-value for H_0^1 than for H_0^2 is consistent with the fact that the magnitude of shift in the intercept is large relative to that of the shift in the slope. The p-value for stationarity of the error variances is 0·1278. So, stationarity of the error variances can hardly be rejected at 10% significance level. Recall that the error variances were not changed.

The Cusum test does not reject H_0 at 5% and 10% significance levels. The Cusum of squares test does reject H_0 at 5% significance level, but does not reject at 1% significance level. Note that the Cusum and the Cusum of squares tests cannot be applied to the individual parameter stationarity testing.

Appendix 1

Local sensitivity analysis for the posterior odds

Assume that the conjugate prior distribution is assigned for $(\beta_1, \beta_2, r_1, r_2)$. The joint prior distribution of β_i and r_i in the ith regime is a multivariate normal-gamma distribution. Specifically,

Table 3. *Results of Bayesian significance test of Quandt's* (1958) *data*; *t, observation number*

t	y_t	x_t	$\pi(\tau \mid Y, X)$	$p_{\delta_\alpha=0\mid Y,X}$	$p_{\delta_\beta=0\mid Y,X}$	$p_{\rho=1\mid Y,X}$
1	3·473	4				
2	11·555	13				
3	5·714	5	0·0088	0·4889	0·2868	0·4075
4	5·710	2	0·0017	0·7412	0·7088	0·5727
5	6·046	6	0·0027	0·5729	0·5877	0·3963
6	7·650	8	0·0046	0·4145	0·4998	0·2464
7	3·140	1	0·0315	0·1258	0·1939	0·2059
8	10·312	12	0·0577	0·0398	0·1319	0·1251
9	13·353	17	0·0805	0·0084	0·1179	0·0639
10	17·197	20	0·0932	0·0037	0·0267	0·0767
11	13·036	15	0·1614	0·0092	0·0305	0·0365
12	8·264	11	0·4957	0·0052	0·0223	0·1469
13	7·612	3	0·0466	0·0247	0·0570	0·3179
14	11·802	14	0·0030	0·3100	0·5659	0·0516
15	12·551	16	0·0053	0·3343	0·7685	0·0457
16	10·296	10	0·0037	0·3380	0·6785	0·1114
17	10·014	7	0·0037	0·8560	0·8677	0·2104
18	15·472	19				
19	15·650	18				
20	9·871	9				
Unconditional p-value				0·0285	0·0634	0·1278

$p_{\delta_\alpha=0\mid Y,X}$, conditional p-value on τ for H_0^1: $\delta_\alpha = 0$; $p_{\delta_\beta=0\mid Y,X}$, conditional p-value on τ for H_0^2: $\delta_\beta = 0$; $p_{\rho=1\mid Y,X}$, conditional p-value on τ for H_{02}: $\rho = 1$.

suppose that the conditional distribution of β_i given r_i is a p-dimensional multivariate normal distribution with the mean vector β_i^*, and precision matrix $r_i T_i^*$ and that the marginal distribution of r_i is a gamma with parameters ν_i^* and λ_i^*. In addition, (β_1, r_1) and (β_2, r_2) are assumed to be independent. Therefore, the hyperparameter set is $\Phi = \{(\beta_i^*, T_i^*, \nu_i^*, \lambda_i^*), i = 1, 2\}$. When $\pi(H_0) = q$, $\pi(H_1) = 1 - q$, and the null parameter set $\Theta_0 = (\beta, \sigma^2)$, the reciprocal of the posterior odds is

$$\Omega_{10} = \frac{1-q}{q}\left\{\frac{(\lambda_2^*)^{\nu_2^*}|T_2^*|^{\frac{1}{2}}}{(2)^{\nu_2^*}\Gamma(\nu_2^*)}\right\} \sum_\tau \Lambda_\tau(\Phi),$$

where

$$\Lambda_\tau(\Phi) = \frac{\Gamma(\frac{1}{2}\tau + \nu_1^*)\Gamma(\frac{1}{2}T - \frac{1}{2}\tau + 2\nu_2^*)|X_1'X_1 + T_1^*|^{-\frac{1}{2}}|X_2'X_2 + T_2^*|^{-\frac{1}{2}} R_1^{-\frac{1}{2}(\tau + 2\nu_1^*)} R_2^{-\frac{1}{2}(T - \tau + 2\nu_2^*)}}{(T-1)\Gamma(\frac{1}{2}T + 2\nu_1^*)|X'X + T_1^*|^{-\frac{1}{2}} R^{-\frac{1}{2}(T + 2\nu_1^*)}},$$

$$R_i = (\hat\beta_i - \beta_i^*)'\{T_i^*(X_i'X_i + T_i^*)^{-1}X_i'X_i\}(\hat\beta - \beta_i^*) + \text{SSE}_i + 2\nu_i^* \quad (i = 1, 2),$$

$$R = (\hat\beta - \beta_1^*)'\{T_1^*(X'X + T_1^*)^{-1}X'X\}(\hat\beta - \beta_1^*) + \text{SSE} + 2\nu_1^*.$$

Then,

$$\frac{d\Omega_{10}}{d\beta_2^*} = K \sum_\tau \{T_2^*(X_2'X_2 + T_2^*)^{-1}X_2'X_2\}\beta_2^*,$$

where K is a constant with respect to β_2^*. The variations in β_2^* affect the posterior feature linearly, and robustness is of concern. It also can be shown that the variations in the other hyperparameters affect Ω_{10} nonlinearly.

Appendix 2

Derivation of the posterior distribution of δ and ρ

Transforming the parameter set $\Theta = (\tau, \beta_1, \beta_2, r_1, r_2)$ into $\Psi = (\tau, \delta, \rho)$, we can form the posterior distribution of Ψ; that is,

$$\begin{aligned}\pi(\Psi \mid Y, X) &= \int_{\beta_2} \int_{r_2} \pi(\tau, \delta + \beta_2, \beta_2, \rho r_2, r_2 \mid Y, X) |r_2| \, dr_2 \, d\beta_2 \\ &\propto \pi(\tau) \rho^{\frac{1}{2}T-1} \Gamma(\tfrac{1}{2}T - \tfrac{1}{2}p) |\rho X_1' X_1 + X_2' X_2|^{-\frac{1}{2}} \\ &\quad \times [\rho \mathrm{SSE}_1 + \mathrm{SSE}_2 + \{\delta - (\hat{\beta}_1 - \hat{\beta}_2)\}' M \{\delta - (\hat{\beta}_1 - \hat{\beta}_2)\}]^{-\frac{1}{2}(T-p)} \\ &= \pi(\delta \mid \tau, \rho, Y, X) \pi(\rho \mid \tau, Y, X) \pi(\tau \mid Y, X).\end{aligned}$$

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