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The Extent of Nonstationarity of Beta

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Abstract. This paper investigates the extent of nonstationarity of beta across the firm size and the beta magnitude by suggesting the sequential parameter stationarity model and estimating change-points of betas. The high-beta firm has shorter stationary interval, which means that its beta changes more frequently than do the low-beta firm's. The firm size, however, does not have a monotonic relation with the length of stationary interval. The small and large firms have relatively shorter stationary interval than do the mid-sized firms. The average length of stationary interval is estimated about five years (exactly 54.19 months). This fact could support the currently widely-used arbitrary 5-year assumption of beta stationarity. The fluctuation of the large firm's beta is more severe than the small firm's, and the high- and low-beta firms have the relatively greater fluctuating betas than do the mid-beta firms. The frequency of detected change-points is found to be positively related to market returns. When the market return is high, the systematic risk changes more frequently, and vice versa.

Key words: Nonstationarity, beta, firm size, change-point

1. Introduction

The beta coefficient in the market model has been widely used as a measure of systematic risk. The implicit assumption behind estimating beta is that the beta or the market model is stationary over time. This stationarity assumption is crucial in investment and portfolio analysis. However, most empirical studies do not support this assumption. Especially, the stationarity of the beta over relatively long period of time is severely questioned. Examples of such empirical studies are performed by Blume (1971, 1975), Levy (1971), Baesel (1974), Altman, Jacquillat, and Levasseur (1974), Fabozzi and Francis (1978), Roenfelt, Griepentrog, and Pflaum (1978), Alexander and Chervany (1980), Sunder (1980), Chen (1981), Bos and Newbold (1984), and many others.

A variety of nonstationarity beta models have been suggested. Among them, the Theil model[1] (Fabozzi and Francis (1978), Bos and Newbold (1980)), the random walk model[2] (Sunder (1980)), a first-order autoregressive model[3] (Ohlsom and Rosenbert (1982)), and a constant beta coefficient model are representative. However, the first three models allow the beta to change too frequently (at every time period). The constant beta coefficient model assumes that betas are stationary over a regular time period, say 5 years, because of its convenience. This model is widely used especially in asset pricing test.

For the constant coefficient beta model, a number of empirical studies has pursued to find the optimal regular interval over which betas are stationary. Blume (1971) suggested cautiously 7-year period, and Gonedes (1973) reported that a 7-year estimation period resulted in more efficient estimator among a 3-year, 5-year, 7-year, 10-year, and 21-year estimation intervals. Five-year period, however, has been widely used in empirical studies such as the univariate or multivariate capital asset pricing model test, since Fama and MacBeth (1973).

The efficient market hypothesis says that stock price reflects quickly the current available information. In other words, the stock return process is affected by the arrival of relevant informaton, and parameters will be changed at a time point (Hereafter I call it the *change-point*). While the announcement of market-wide information might cause equal change-points across the firm, the announcement of firm-specific events might cause different change-points since the firm-specific events occur at different times across the firm. The interval between change-points is stochastic, therefore, the regular interval assumption might result in inaccurate estimation of the beta.

The essential work for the more precise estimation of beta is to estimate the unknown change-points at which the betas change. No one (to my knowledge) has ever sought to discover at what intervals systematic risks change. But, Kim and Kon (1992) estimated the mean and variance change-points of the Dow Jones Industry 30 stocks and market indexes for formulating the daily stock return distrinutional model, and concluded that the sequential normal mixture model fits better the stock returns than Student t, Poisson jump process, and the generalized normal mixtures. This paper suggests the sequential parameter nonstationarity model in which the change-points of beta are estimated *sequentially* by a Bayesian approach. Based on the detected change-points, I investigate whether there are distinctive characteristics in the beta estimate and change-point behaviors across the beta magnitude and the firm size, and whether the *ad hoc* 5-year beta estimation method is supported by estimating the overall average interval of stationary betas. This paper does not pursue to find the cause of changing betas.

This paper is organized as follows: in the second section, the sequential parameter nonstationarity model is formulated and the detection method of the change-points is provided. The data and empirical results are presented in the third and fourth sections, respectively. Concluding remarks are given in the last section.

2. Statistical Methodology

2.1. Formulation of A Sequential Parameter Nonstationarity

Under a given set of assumptions, the *ex ante* expected return on an asset will be determined in eqiulibrium by the Sharpe capital asset pricing model (CAPM) by the

$$E(R_{it}) = R_{ft}(1 - \beta_i) + \beta_i E(R_{mt}). \tag{1}$$

In this expression, $E(R_{it})$ is the expected return on asset i in period t, R_{ft} is the return on the risk free asset, $E(R_{mt})$ is the expected return on the market portfolio, and β_i is the asset i's systematic risk.

The *ex ante* equilibrium relation in (1) can be replaced in empiricial studies by an equation for *ex post* returns, given by

$$R_{it} = R_{ft}(1 - \beta_i) + \beta_i R_{mt} + \epsilon_{it}, \tag{2}$$

where R_{it} and R_{mt} denote the *ex post* return on asset i and on the market, respectively, at time t, and ϵ_{it} is independently normally distributed with mean 0 and variance σ_i^2. The development of (2) depends on the equilibrium relation that the first term on the righthand side equals $R_{ft}(1 - \beta_i)$. Without any constraint on the first term, one obtains the market model,

$$R_{it} = \alpha_i + \beta_i R_{mt} + \epsilon_{it}.$$

This model assumes that the slope, intercept, and error variance are constant over given time period. This is a strong assumption if the time period is long.

The specification of the nonstationarity in the sequential parameter nonstationarity model is that the structure of the market model is stationary up to a change-point, and thereafter, is stationary from the previous change-point to the next change-point, and so forth. The formulation of the sequential parameter nonstationarity model is as follows:

$$\begin{aligned} &\textit{Regime } 1 : R_t = \alpha_1 + \beta_1 R_{mt} + \epsilon_{1t}, && t = 1, 2, \ldots, \tau_1 \\ &\textit{Regime } 2 : R_t = \alpha_2 + \beta_2 R_{mt} + \epsilon_{2t}, && t = \tau_1 + 1, \ldots, \tau_2 \\ &\textit{Regime } K : R_t = \alpha_K + \beta_K R_{mt} + \epsilon_{Kt}, && t = \tau_{K-1} + 1, \ldots, \tau_K (\equiv T) \end{aligned} \quad (3)$$

where ϵ_{kt} is identically and independently normally distributed with mean 0 and variance σ_k^2. Note that parameters $\tau_1, \ldots, \tau_{K-1}$ define the unknown change-points *sequentially*.

It is natural to assume that all parameters are changed when the regime is changed. Specifically, the beta will be different when the risk structure is changed, and vice versa. So, I assume the non-constant error variances.

The next section describes how to estimate the change-points vector $\Upsilon = (\tau_1, \ldots, \tau_K)$ and then the betas $(\beta_1, \ldots, \beta_K)$. The estimated betas (stationarity-adjusted), therefore, are conditional on the detected change-points.

2.2. A Detection Procedure for Change-Points

The major task of the estimation procedure in the sequential parameter nonstationarity model is how to detect change-points vector, Υ. One of the detection methods is a Bayesian approach, which outperforms the maximum likelihood methods, even if the diffuse prior is applied (Kim (1991)). In order to detect change-points within the Bayesian framework, one needs the joint posterior density function of Υ.

When one applies the diffuse prior for parameter set $\Theta = \{\alpha_1, \ldots, \alpha_K, \beta_1, \ldots, \beta_K, \tau_1, \ldots, \tau_{K-1}\}$, the joint posterior distribution of the change-points can be derived as

$$\pi(\Upsilon \mid \mathbf{R}) \propto \pi(\Upsilon) \prod_{k=1}^{K} \left\{ (\tau_k - \tau_{k-1})^{-1/2} \, \Gamma \left[\frac{\tau_k - \tau_{k-1} - 2}{2} \right] (S\,S\,E_k)^{-(\tau_k - \tau_{k-1} - 2)/2} \right\}, \quad (4)$$

where

$$S\,S\,E_k = \sum_{t=\tau_{k-1}+1}^{\tau_k} \{R_t - (\hat{\alpha}_k + \hat{\beta}_k R_{mt})\}^2,$$

$\pi(\Upsilon)$ is the joint prior distribution of the change points, $\Gamma(\cdot)$ is a gamma function, and $\hat{\alpha}_k$ and $\hat{\beta}_k$ are ordinary least squares estimators in the k-th regime. The symbol $\propto$ indicates *proportional to*. Hereafter, the function $\pi(\Theta)$ and $\pi(\Theta \mid \cdot)$ denote the prior and the posterior distribution of the parameter Θ, respectively. Especially, when there is only one change-point (τ), the posterior distribution of the change-point is

$$\pi(\tau \mid \mathbf{R}) \propto \pi(\tau)\, \{\tau(T - \tau)\}^{-1/2}\, \Gamma\left(\frac{\tau - 2}{2}\right)\, \Gamma\left(\frac{T - \tau - 2}{2}\right) (S\,S\,E_1)^{-(\tau-2)/2}\, (S\,S\,E_2)^{-(T-\tau-2)/2} \quad (5)$$

There are two ways to detect the change-points vector, *simultaneously* and *sequentially.* The simultaneous detection method is that if the number of regimes K is known, the change-points vector with the largest posterior probability (i.e., posterior mode of $\pi(\Upsilon \mid \mathbf{R})$ or generalized maximum likelihood estimation) is chosen as the estimates of change points. In reality, however, the number of regimes is not explicitly known. Despite the number of regimes is known, there will be situations in which the simultaneous detection of change-points is computationally difficult. For example, if the sample size is T and the number of regimes is K, approximately $\binom{T}{K-1}$ computations of (4) are needed. If the number of observations is $T = 500$ and the number of regimes is $K = 10$, then approximately 2.458×10^{20} times computations of posterior probability of (4) are needed. It is a serious barrier in the simultaneous detection method. On the other hand, the sequential detection method, which is in fact an ad hoc method, could resolve these computational difficulties and moreover estimate the number of regimes K.

The sequential detection procedure of change-points is described as follows: The test on the stationarity of parameters is applied to the first several data points. This test is repeated as each data point is added until the null hypothesis that the parameters are stationary over the given sample period is rejected at an assigned significance level. If the null hypothesis is rejected, it is reasonable to consider that there is a structural shift over the sample period, that is, there exists one change-point. Now, in order to detect the change-point over the period, one applies (5) and chooses the posterior mode as an estimate of the change-point. The above procedure is repeated using the first data point following the previously detected change-point as the initial data point, until all data are scanned. Then one could estimate the total change-points vector and the number of regimes.

A testing procedure of the stationarity of parameters is needed for the sequential detection method. The Bayesian significance test using the highest posterior density (HPD), interval[4] (hereafter, *HPD interval test*) is employed in this paper (see in detail Lindley (1965), Box and Tiao (1973), and Kim (1991)).[5] In addition to its power, another advantage is its ability to determine which parameter among alpha, beta, and error variance is more responsible for structural shift of the market model. This test may provide some information on this matter.

The HPD interval test is described briefly as follows: we suppose $K = 2$ over the given testing period, that is, there is an unknown change-point. The parameters are $(\alpha_1, \beta_1, \sigma_1^2)$ for the first regime, and $(\alpha_2, \beta_2, \sigma_2^2)$ for the second regime, and the change-point τ. We have the null hypothesis that the market model over the testing period (say, 1 to n) is stationary, that is,

$$H_0 : (\delta_\alpha, \delta_\beta, \rho) = \mathbf{0},$$

where

$$\delta_\alpha = \alpha_1 - \alpha_2,$$
$$\delta_\beta = \beta_1 - \beta_2,$$
$$\rho = \sigma_2^2/\sigma_1^2.$$

One can divide the null hypothesis H_0 into three sub-nulls

$$H_{01} : \delta_\alpha = 0, \qquad H_{02} : \delta_\beta = 0, \qquad \text{and} \qquad H_{03} : \rho = 1,$$

and reject H_0 if one of these three sub-nulls is rejected. In other words, if one of p-values for H_{01}, H_{02}, and H_{03} is smaller than an assigned significance level, then the null H_0 is rejected at the given significance level.[6] H_{01}, H_{02}, and H_{03} are subnulls for stationarity of intercept, slope, and error variance in the market model, respectively.

The p-value for H_{01} is computed as follows: the conditional p-value for H_{01} given ρ and τ is

$$p_{\delta_\alpha=0|_{\rho,\tau}} = E_{\bar{C}_{\delta_\alpha}} [\pi(\delta_\alpha \mid \rho, \tau, \mathbf{R})],$$

where $E_{\bar{C}_{\delta_\alpha}} [\pi(\cdot)]$ indicates the area of a posterior distribution $\pi(\cdot)$ over the region $\bar{C}_{\delta_\alpha}$ which is the complement of C_{δ_α}, $(1 - \alpha)$ credible set of δ_α (HPD interval), and $\pi(\delta_\alpha \mid \rho, \tau, \mathbf{R})$ is the conditional posterior distribution of δ_α given ρ and τ. C_{δ_α} is defined as

$$C_{\delta_\alpha} = \{\delta_\alpha : \pi_{\alpha/2}(\delta_\alpha \mid \rho, \tau, \mathbf{R}) \le \delta_\alpha \le \pi_{1-\alpha/2}(\delta_\alpha \mid \rho, \tau, \mathbf{R})\},$$

where $\pi_{\alpha/2}(\delta_\alpha \mid \cdot)$ indicates the $\alpha/2$-th quantile of the conditional posterior distribution of δ_α. The conditional posterior distribution of δ_α given ρ and τ is described as

$$\pi(\delta_\alpha \mid \rho, \tau, \mathbf{R}) \propto \left(1 + \frac{M_\alpha\,(\delta_\alpha - \hat{\delta}_\alpha)^2}{\rho\, SSE_1 + SSE_2} \right)^{-(n-3)/2},$$

where

$$\hat{\delta}_\alpha = \hat{\alpha}_1 - \hat{\alpha}_2,$$

$$M_\alpha = \frac{\rho\tau(n - \tau)Sxx_1\ Sxx_2}{\rho\ \tau\ Sxx_1\ \Sigma_{t=1}^{\tau}\ R_{mt}^2 + (n - \tau)\ Sxx_2\ \Sigma_{t=\tau+1}^{n}\ R_{mt}^2}$$

$$Sxx_k = \sum_{k-th} (R_{mt} - \bar{R}_m)^2, \qquad i = 1, 2,$$

$$S\ S\ E_k = \sum_{k\text{-}th} [R_{it} - (\hat{\alpha}_k + \hat{\beta}_k R_{mt})]^2, \qquad i = 1, 2,$$

where $\hat{\alpha}_k$ and $\hat{\beta}_k$ are the ordinary least square estimates. In fact, the conditional posterior distribution of δ_α is the Student t distribution with location $\hat{\delta}_\alpha$, precision $(n - 4)\ M_\alpha / (\rho\ S\ S\ E_1 + S\ S\ E_2)$, and degrees of freedom $(n - 4)$.

The *unconditional* p-value for H_{01}, therefore, is defined as

$$p_{\delta_\alpha=0} = \sum_\tau \left\{ \int_\rho E_{\bar{C}_{\delta_\alpha}} [\pi(\delta_\alpha \mid \rho, \tau, \mathbf{R})]\ \pi(\rho \mid \tau, \mathbf{R})\ d\ \rho \right\} \pi(\tau \mid \mathbf{R}), \tag{6}$$

where $\pi(\rho \mid \tau, \mathbf{R})$ is the conditional posterior distribution of ρ given τ. The unconditional p-value is the weighted average of the conditional p-value with the weight determined by the $\pi(\rho \mid \tau, \mathbf{R})$ and $\pi(\tau \mid \mathbf{R})$.

Similarly, the p-value for H_{02} is defined as

$$p_{\delta_\beta=0} = \sum_\tau \left\{ \int_\rho E_{\bar{C}_{\delta_\beta}} [\pi(\delta_\beta \mid \rho, \tau, \mathbf{R})]\ \pi(\rho \mid \tau, \mathbf{R})\ d\ \rho \right\} \pi(\tau \mid \mathbf{R}), \tag{7}$$

where $\pi(\delta_\beta \mid \rho, \tau, \mathbf{R})$ is the Student t distribution described as

$$\pi(\delta_\beta \mid \rho, \tau, \mathbf{R}) \propto \left(1 + \frac{M_\beta\ (\delta_\beta - \hat{\delta}_\beta)^2}{\rho\ S\ S\ E_1 + S\ S\ E_2} \right)^{-(n-3)/2},$$

where

$$\hat{\delta}_\beta = \hat{\beta}_1 - \hat{\beta}_2,$$

$$M_\beta = \frac{\rho\ Sxx_1\ Sxx_2}{\rho\ Sxx_1 + Sxx_2}.$$

The p-value for the subnull H_{03} is also defined as

$$p_{\rho=1} = \sum_\tau E_{\bar{C}_\rho}[\pi(\rho \mid \tau, \mathbf{R})]\ \pi(\tau \mid \mathbf{R}), \tag{8}$$

where the transformed variable $\rho'\{=(\hat{\sigma}_2^2/\hat{\sigma}_1^2)\rho\}$ is an F-variate with $(\tau - 2, n - \tau - 2)$ degrees of freedom, and $\hat{\sigma}_k^2$ is the estimated error variance in the k-th regime. This p-value is also the weighted average of the conditional p-value by the F-test with the weight by $\pi(\tau \mid \mathbf{R})$.

With the computed p-values through (6), (7), and (8), we could determine which parameter is more severely nonstationary. When the p-value for H_{01} is smaller, but the p-values for H_{02} and H_{03} is larger than an assigned significance level, the alpha (intercept) change might be more responsible for the structural shift in the model than is the beta or error variance change. This does not mean, however, that the beta or error variance is stationary. It is possible that the nonstationarity of the beta or error variance is not severe enough to be detected by the testing device. Throughout this paper, 5% significance level is used.

For the derivation of the conditional posterior distribution of δ_α, δ_β, and ρ, and the posterior distribution of Υ, refer to Kim (1991).

3. Data

I gather return and fir-size data from the Center for Research in Security Prices (CRSP) monthly return tape at the University of Chicago. Three-month U.S. Treasury bill monthly returns are obtained from Ibbotson (1990). At the end of each year, firms on the NYSE or AMEX that have existed for at least five years from 1926 to 1990 are ranked by their market values for the following year and are assigned into one of 10 firm size portfolios based on their relative position in their market values.[7] Within a firm size portfolio, all available stocks are ranked by their stationarity-adjusted betas for the following year and are assigned into one of 10 beta portfolios. A firm is therefore assigned into one of 100 portfolios. The portfolio returns are then computed by combining monthly returns of all individual securities within the portfolio with equal weights. The compositions of the portfolios are rebalanced annually. Table 1 provides each portfolio's equal-weighted average returns.

Table 1. Average portfolio's monthly returns from 1926 to 1990.

Size-sorted Portfolios	Beta-sorted Portfolios										
	Lowest	2	3	4	5	6	7	8	9	Highest	Ave
smallest	0.0087	0.0180	0.0197	0.0185	0.0215	0.0233	0.0113	0.0242	0.0260	0.0321	0.0203
2	0.0137	0.0128	0.0140	0.0117	0.0140	0.0138	0.0176	0.0124	0.0121	0.0201	0.0142
3	0.0126	0.0114	0.0113	0.0146	0.0107	0.0124	0.0132	0.0121	0.0145	0.0141	0.0127
4	0.0108	0.0113	0.0114	0.0112	0.0127	0.0152	0.0153	0.0133	0.0138	0.0147	0.0130
5	0.0100	0.0114	0.0099	0.0110	0.0131	0.0129	0.0115	0.0130	0.0116	0.0151	0.0120
6	0.0105	0.0105	0.0111	0.0104	0.0121	0.0134	0.0100	0.0137	0.0118	0.0161	0.0119
7	0.0101	0.0094	0.0110	0.0122	0.0126	0.0122	0.0094	0.0136	0.0132	0.0134	0.0117
8	0.0105	0.0087	0.0105	0.0110	0.0101	0.0110	0.0124	0.0099	0.0114	0.0091	0.0105
9	0.0102	0.0092	0.0093	0.0103	0.0116	0.0119	0.0110	0.0114	0.0118	0.0117	0.0108
largest	0.0082	0.0088	0.0086	0.0089	0.0078	0.0081	0.0089	0.0101	0.0097	0.0114	0.0090
Ave	0.0105	0.0111	0.0117	0.0120	0.0126	0.0134	0.0121	0.0134	0.0136	0.0158	0.0126

4. Empirical Results

4.1. Relation Between the Length of Stationary Interval and Firm Size and Beta Magnitude

At a time point t, the length of stationary interval is defined as

$$I_t = \tau_k - \tau_{k-1},$$

where $\tau_{k-1} < t \leq \tau_k$. Likewise, the stationarity-adjusted beta at time t is defined as

$$\tilde{\beta}_t = \tilde{\beta}_k,$$

where $\tilde{\beta}_k$ is the ordinary least square estimate in the k-th regime.

Each portfolio's length of stationary interval and beta are then computed by combining corresponding values of all individual firms within the portfolio with equal weights. Each portfolio's one-regime betas, assuming whole sample period is stationary, is reported in Panel A, and the stationarity-adjusted betas after estimating each regime's beta over stationary interval is presented in Panel B of table 2. Both betas are monotonic across the firm size and beta magnitude. However, the spread of the stationarity-adjusted betas (1.890), which is the difference between the largest and smallest betas, is greater than that of the one-regime betas (1.632).

Table 3 presents the average length of stationary interval of betas and its standrad error. These average values are statistically very significant. The mganitude of beta has an inverse monotonic relationship with the length of stationary interval. In other words, the high beta firms have shorter stationary interval of betas, while the low beta firms have longer interval. These findings indicate that the high beta firms' betas are more unstable, and their return structure is more sensitive to relevant information. The firm size, however, does not have a monotonic relation with the length of stationary intervals. The small and large firms have relatively shorter stationary intervals than do the mid-sized firms. These facts imply that the errors-in-variable problem in the second-stage asset pricing testing procedure might be more severe in the high-beta and mid-sized firms. The length of stationary interval of betas should be considered in order to reduce the errors-in-variable problem. However, this paper does not investigate this issue.

Another empirically interesting thing found is that the overall average length of stationary interval is about five years (54.19 months). This fact could support the currently widely-used arbitrary 5-year assumption of beta stationarity. The shorter interval (say, 3 years), however, should be used for estimating smaller firm's beta. Even though I construct portfolios first by the beta magnitude and then by firm size within the beta-sorted portfolio, the above findings are not changed.

A distinctive feature in the frequency of detected change-points across the month of the year is not found. Specifically, the change-points detected at January is not different from those at the other months. The frequency of detected change-points at a specific time t is defined here as the total number of change-points detected at t of all available firms divided by the total number of available firms. It is actually the average number of change-points per firm.

Table 2. Estimated portfolio's betas from 1926 to 1990.

Size-sorted Portfolios	Beta-sorted Portfolios										
	Lowest	2	3	4	5	6	7	8	9	Highest	Ave
*Panel A: One-regime Betas**											
smallest	0.786	0.989	1.186	1.217	1.442	1.475	1.430	1.641	1.719	2.022	1.391
2	0.701	0.891	0.905	1.079	1.199	1.263	1.407	1.568	1.553	1.752	1.232
3	0.669	0.630	0.835	0.948	1.008	1.124	1.303	1.312	1.470	1.659	1.096
4	0.546	0.697	0.742	0.974	0.935	1.101	1.251	1.276	1.365	1.503	1.039
5	0.596	0.606	0.708	0.837	0.936	1.004	1.135	1.221	1.298	1.481	0.982
6	0.492	0.501	0.652	0.733	0.877	0.985	1.088	1.201	1.301	1.477	0.931
7	0.510	0.503	0.624	0.722	0.772	0.907	1.030	1.075	1.220	1.396	0.876
8	0.470	0.402	0.539	0.620	0.743	0.839	0.932	1.020	1.144	1.272	0.798
9	0.382	0.456	0.554	0.672	0.696	0.832	0.888	0.973	1.012	1.179	0.764
largest	0.390	0.406	0.475	0.503	0.585	0.637	0.702	0.792	0.900	1.015	0.640
Ave	0.554	0.608	0.722	0.830	0.919	1.017	1.117	1.208	1.298	1.476	0.975
*Panel B: Stationarity-adjusted Betas***											
smallest	0.695	0.759	0.981	1.018	1.172	1.320	1.383	1.574	1.761	2.297	1.296
2	0.647	0.688	0.819	0.922	1.071	1.203	1.301	1.450	1.600	1.960	1.166
3	0.604	0.596	0.760	0.901	0.991	1.113	1.241	1.387	1.531	1.846	1.097
4	0.521	0.571	0.686	0.828	0.970	1.062	1.161	1.307	1.476	1.707	1.029
5	0.495	0.536	0.698	0.831	0.955	1.014	1.155	1.267	1.429	1.717	1.010
6	0.526	0.505	0.664	0.785	0.892	0.991	1.114	1.208	1.358	1.633	0.967
7	0.484	0.479	0.620	0.737	0.847	0.935	1.022	1.159	1.312	1.572	0.917
8	0.451	0.473	0.590	0.697	0.803	0.887	0.988	1.099	1.255	1.459	0.870
9	0.421	0.453	0.560	0.670	0.762	0.825	0.909	0.983	1.098	1.337	0.802
largest	0.407	0.427	0.511	0.596	0.645	0.735	0.808	0.890	0.974	1.171	0.716
Ave	0.525	0.549	0.689	0.798	0.911	1.008	1.108	1.232	1.380	1.670	0.987

*Estimating betas by assuming whole sample period is stationary, i.e., one regime.
**Estimating each regime's stationary betas.

The frequency of detected change-points over time (equivalently, the inverse of the length of stationary interval), however, has close positive relation to the market returns (the correlation coefficient is 0.257, and its t-value is 2.11), but negative relation to the three-month Treasury bills monthly returns (the correlation coefficient is -0.358, and its t-value is 3.04). The correlation coefficient between the frequency of detected change-points and the market risk premium is 0.293 and its t-value is 2.43. The rank correlation coefficient is 0.201 and its p-value is 0.0091. Figure 1 shows the co-movement of the yearly frequency of detected change-points and yearly market returns. When the market returns are high, the systematic risk changes more frequently, and in bearish market, the systematic risk changes less frequently. Especially, the tuning points of two variables in this figure are similar shaped and almost coincident. It is interesting to mention that the risk structure has been changed more frequently when the market returns are high. One interpretation is, therefore, that the interval might be another proxy for *risk* which is not captured by the market model.

Table 3. Average length of stationary change-points interval (in months).

Size-sorted Portfolios	Beta-sorted Portfolios Lowest	2	3	4	5	6	7	8	9	Highest	Ave
smallest	27.28	29.73	32.29	30.83	31.01	35.13	37.18	36.19	35.91	37.40	33.29
2	42.02	47.30	46.97	51.97	51.12	52.89	53.34	56.17	57.92	58.24	51.79
3	46.62	50.66	53.10	53.21	55.15	54.61	55.28	64.75	65.00	65.97	56.43
4	52.73	56.06	56.43	54.35	58.97	58.82	60.11	66.94	66.43	68.49	59.93
5	54.71	60.44	58.71	58.54	59.73	65.77	66.95	65.42	67.76	68.47	62.65
6	54.12	53.42	58.31	58.37	62.66	63.88	63.06	66.08	73.63	74.75	62.83
7	52.96	55.39	56.07	61.33	61.23	60.78	63.73	66.51	69.89	71.50	61.94
8	48.89	53.05	55.22	57.30	61.33	57.54	61.48	66.57	68.89	69.50	59.98
9	46.68	48.41	48.37	50.87	53.35	52.54	57.99	61.28	61.48	62.45	54.34
largest	33.31	33.16	33.92	38.10	37.35	38.81	38.47	40.65	45.13	46.64	38.55
Ave	45.93	48.76	49.94	51.49	53.19	54.08	55.76	59.06	61.20	62.34	54.19
					(standard errors)						
smallest	0.3654	0.4532	0.5254	0.6168	0.6012	0.5707	0.5220	0.5973	0.5143	0.4710	
2	0.5733	0.5821	0.5362	0.5468	0.6005	0.5033	0.5160	0.5484	0.5550	0.4000	
3	0.5102	0.5480	0.5843	0.5777	0.5966	0.6565	0.5759	0.5023	0.5126	0.4308	
4	0.4543	0.5649	0.5785	0.5755	0.6826	0.6528	0.5950	0.4900	0.5333	0.4524	
5	0.4537	0.6479	0.6494	0.6627	0.6480	0.7167	0.5508	0.5559	0.5069	0.3984	
6	0.5893	0.6008	0.6642	0.6569	0.7172	0.7058	0.7252	0.7002	0.5998	0.4535	
7	0.5216	0.6516	0.7233	0.6391	0.7465	0.7504	0.7185	0.6227	0.6033	0.4748	
8	0.5171	0.6975	0.8988	0.8262	0.7364	0.6550	0.7942	0.6794	0.6024	0.4551	
9	0.6387	0.7699	0.8175	0.7398	0.6825	1.0318	0.9732	0.7543	0.7597	0.5325	
largest	0.5791	0.7875	0.7411	0.8520	0.8671	0.8446	0.7823	0.7656	0.7295	0.5923	

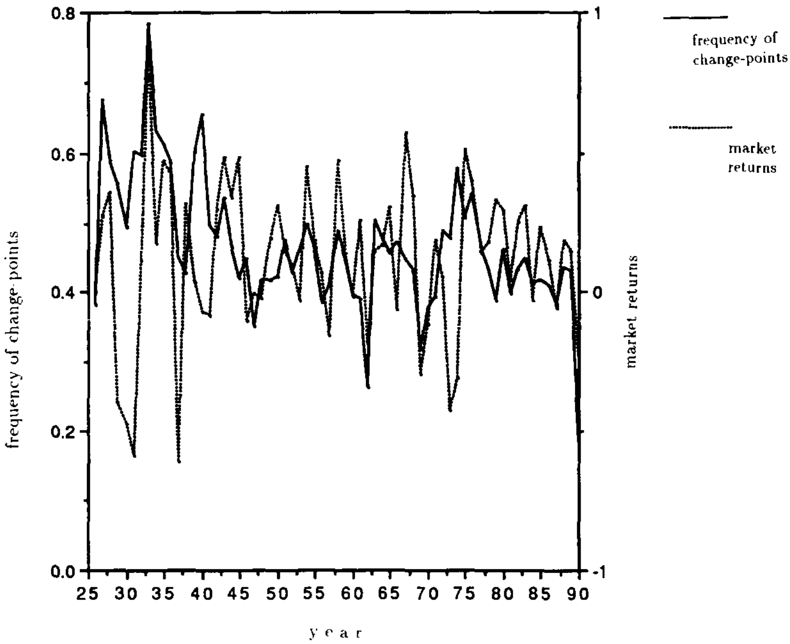

Figure 1. The yearly frequency of detected change-points and the CRSP equal-weighted yearly market returns.

Treynor and Mazuy (1966) found that mutual fund managers did not reduce the fund's beta in bearish market and increase it in bullish market in order to earn higher risk-adjusted returns. It is unclear from the above findings, however, that betas are shifted upward in bullish market and downward in bearish market. More investigation is needed to study fund manager's timing ability. The results in figure 1 can also say that betas are more frequently shifted when the market returns increase (possibly, in the market transition from the bearish to the bullish) rather than when the market returns decrease. These findings are consistent with Fabozzi and Francis (1979a, 1979b) who found that betas tend to experience significant shifts during the bullish-bearish market period.

4.2. Variability of Betas

It would be interesting to investigate whether there is a distinctive feature in statistical behavior of systematic risk across the firm size and the beta magnitude. In order to measure this statistical behavior, one uses the following measures: mean relative bias, absolute mean relative bias, and mean relative squared error. The mean relative bias (*MRB*) of the stationarity-adjusted betas is defined as

$$MRB = \frac{1}{T} \sum_{t=1}^{T} \frac{\hat{\beta}_{pt} - \hat{\beta}_o}{\hat{\beta}_o},$$

where $\hat{\beta}_{pt}$ is the portfolio p's stationarity-adjusted beta at time t, $\hat{\beta}_o$ is the portfolio p's one-regime beta, and T is the whole sample period. Likewise, the absolute mean relative bias (*AMRB*) and the mean relative squared error (*MRSE*) are defined as

$$AMRB = \frac{1}{T} \sum_{t=1}^{T} \left| \frac{\hat{\beta}_{pt} - \hat{\beta}_o}{\hat{\beta}_o} \right|,$$

and

$$MRSE = \frac{1}{T} \sum_{t=1}^{T} \left(\frac{\hat{\beta}_{pt} - \hat{\beta}_o}{\hat{\beta}_o} \right)^2,$$

respectively.

The three variability measures of estimated betas are presented in Panel A, B, and C of table 4. The stationarity-adjusted betas for large-sized and low-beta firms are relatively negatively biased, i.e., are smaller than one-regime betas. On the other hand, the betas for small-sized and high-beta firms are larger than one-regime betas. The remarkable thing is that the absolute mean relative biases and the mean relative squared errors for small-sized and high-beta firms are larger than that for large-sized and low-beta firms. Especially, the magnitude of these measures are monotonically decreasing with the firm size.

Table 4. Variability measures of estimated stationary betas.

Size-sorted Portfolios	Beta-sorted Portfolios: Lowest	2	3	4	5	6	7	8	9	Highest	Ave
*Panel A: Mean Relative Bias (MRB)**											
smallest	0.032	0.030	0.020	0.028	0.048	0.075	0.097	0.089	0.118	0.088	0.063
2	0.034	0.063	0.017	0.027	0.060	0.065	0.104	0.113	0.135	0.138	0.076
3	0.037	0.038	0.018	0.018	0.034	0.061	0.077	0.108	0.089	0.136	0.062
4	0.014	0.027	-0.009	0.034	0.042	0.043	0.064	0.066	0.087	0.131	0.050
5	0.025	-0.035	0.010	0.021	-0.006	0.037	0.064	0.078	0.083	0.144	0.042
6	-0.017	-0.021	0.019	0.010	-0.004	0.031	0.063	0.060	0.069	0.097	0.030
7	-0.011	-0.035	-0.001	0.026	-0.010	0.007	0.029	0.036	0.081	0.100	0.022
8	-0.037	-0.017	-0.012	-0.005	0.025	0.038	0.010	0.055	0.041	0.090	0.018
9	-0.008	-0.043	-0.014	-0.035	0.040	0.027	0.031	0.035	0.041	0.073	0.014
largest	0.057	0.025	0.010	0.040	0.030	0.068	0.064	0.066	0.080	0.144	0.058
Ave	0.012	0.003	0.005	0.016	0.026	0.045	0.060	0.071	0.082	0.114	0.043
*Panel B: Absolute Mean Relative Bias (AMRB)***											
smallest	1.070	0.960	1.057	1.012	1.088	1.054	1.021	1.153	1.174	1.169	1.076
2	0.935	0.955	0.942	0.959	0.925	0.958	0.979	1.022	1.086	0.992	0.975
3	0.916	0.891	0.872	0.874	0.865	0.849	0.882	0.920	1.013	0.930	0.901
4	0.865	0.815	0.828	0.820	0.835	0.807	0.809	0.859	0.854	0.898	0.839
5	0.870	0.794	0.794	0.727	0.780	0.778	0.772	0.805	0.820	0.877	0.802
6	0.848	0.764	0.771	0.738	0.724	0.747	0.745	0.757	0.782	0.852	0.791
7	0.821	0.746	0.749	0.723	0.727	0.735	0.714	0.741	0.779	0.828	0.756
8	0.790	0.727	0.718	0.704	0.724	0.732	0.697	0.735	0.751	0.818	0.739
9	0.802	0.718	0.686	0.697	0.718	0.689	0.693	0.702	0.719	0.808	0.723
largest	0.781	0.714	0.672	0.682	0.702	0.675	0.671	0.690	0.706	0.785	0.719
Ave	0.870	0.808	0.809	0.794	0.809	0.803	0.798	0.838	0.868	0.897	0.829
*Panel C: Mean Relative Squared Error (MRSE)****											
smallest	5.076	4.076	5.252	4.085	4.642	3.673	4.669	7.663	7.087	8.344	5.457
2	3.036	3.386	3.414	3.581	2.953	3.475	2.837	3.099	5.381	4.031	3.519
3	2.870	2.843	2.670	2.732	2.498	2.375	2.654	2.549	5.333	2.833	2.936
4	2.665	2.673	2.311	2.351	2.548	2.303	2.115	2.137	2.063	2.656	2.382
5	2.733	2.528	2.433	2.197	2.186	2.066	1.916	2.011	2.000	2.375	2.209
6	2.495	2.248	2.122	2.006	1.956	1.742	1.903	1.922	1.961	2.097	2.018
7	2.334	2.187	2.072	1.866	1.861	1.709	1.806	1.878	1.921	2.092	1.973
8	2.279	1.796	1.740	1.615	1.774	1.633	1.400	1.792	1.822	2.888	1.874
9	2.212	1.838	1.537	1.655	1.718	1.556	1.500	1.513	1.515	2.792	1.784
largest	2.055	1.631	1.511	1.571	1.632	1.535	1.468	1.505	1.512	2.551	1.697
Ave	2.776	2.521	2.506	2.366	2.377	2.207	2.227	2.607	3.060	3.266	2.591

*Mean relative bias = $(1/T)\ \Sigma_{t=1}^{T}\{(\hat{\beta}_{pt} - \hat{\beta}_o) / \hat{\beta}_o\}$, where $\hat{\beta}_{pt}$ is the portfolio p's estimated beta at time t over the detected stationary interval, $\hat{\beta}_o$ is the estimated overall beta assuming whole sample period is stationary, and T is the whole sample period.

**Absolute mean relative bias = $(1/T)\ \Sigma_{t=1}^{T}\ |\ (\hat{\beta}_{pt} - \hat{\beta}_o) / \hat{\beta}_o\ |$

***Mean relative squared error = $(1/T)\ \Sigma_{t=1}^{T}\{(\hat{\beta}_{pt} - \hat{\beta}_o) / \hat{\beta}_o\}^2$

As the firm size decreases, the fluctuation of betas becomes greater. The fluctuation of the beta-sorted portfolio's stationarity-adjusted betas, however, are not monotonic, but U-shaped. Put differently, the high- and low-beta firms have the more fluctuating betas than do the mid-beta firms.

5. Conclusion

This paper has investigated the extent of nonstationarity of betas across the firm size and the beta magnitude. In order to do this, the sequential parameter nonstationarity model is employed with estimating the change-points at which the betas change.

The length of the stationary interval of beta is inversely related to the magnitude of betas, which means that the high-beta firms have shorter stationary interval and the low-beta firm's stationary interval is longer. In other words, the high-beta firm's beta is more nonstationary. The firm size, however, does not have a monotonic relation to the length of stationary interval. The small and large firms have relatively shorter stationary interval than do the mid-sized firms.

Another empirically interesting finding is that the average length of stationary interval is estimated to be about five years (exactly 54.19 months) This fact could support the currently widely-used arbitrary 5-year assumption of beta stationarity.

Although the frequency of detected change-points is not found to be different across the month of the year, it has close positive relation to the market returns. When the market returns are high, the systematic risk changes more frequently, and vice versa.

The other remarkable thing found is the variability of betas over time. The large-sized firm's beta (stationarity-adjusted) fluctuates more than the small-sized firm's. The relation between the beta magnitude and the extent of fluctuation, however, is not found monotonic. The high-beta and low-beta firms have the relatively greater fluctuating betas than do the mid-beta firms.

Notes

1. The Theil model states that

$$\beta_{it} = \beta_i + \epsilon_{it},$$

where ϵ_{it} is a normally distributed random error term with mean 0.

2. The random walk model says that

$$\beta_{it} = \beta_{it-1} + \epsilon_{it}.$$

3. A first-order autoregression model says that

$$\beta_{it} = \rho\beta_{it-1} + (1 - \rho)\bar{\beta}_i + \epsilon_{it}.$$

where ρ is the autocorrelation coefficient, $\bar{\beta}_i$ is the mean of β_{it}.

4. The HPD interval is defined as follows: Let $\pi(\theta \mid data)$ be a posterior density function of θ. A region C in the parameter space of θ is called an HPD interval of content $(1 - \alpha)$ if $Pr\{\theta \in C \mid data\} = 1 - \alpha$ and, for $\theta_1 \in C$ and $\theta_2 \notin C$, $\pi(\theta_1 \mid data) \geq \pi(\theta_2 \mid data)$. The HPD interval test is different from the traditional Bayes testing procedure carried out within a posterior odds framework. The typical Bayes test presumes that the null value of a target parameter is believed to be stronger than any other value near the null value, a presumption which is sometimes true, but in many applications unsupported. Lindley (1965, p. 61) suggested a Bayesian test of significance by the HPD interval, and emphasized that this type of significance test is appropriate only for circumstances in which prior knowledge of the target parameter is vague or diffuse, and one of the hypotheses is a single point. This method is similar to the sampling theory approach of rejecting a null hypothesis when the hypothesized value for a parameter falls outside a confidence interval. The confidence interval corresponds to the HPD interval in the HPD interval test. If the posterior distribution of a target parameter is concentrated on the hypothesized value of the parameter, then it is hard to reject the null hypothesis. The HPD interval

test is especially useful when the information on target parameters is diffuse. In fact, since we have no information on the stationarity of alpha, beta, and error variance in the market model *a priori*, the HPD interval test is appropriate.

5. In linear regression models, the HPD interval test on the stationarity of parameters has been shown to have stronger testing power than conventional non-Bayesian techniques such as the cusum and cusum of squares.
6. Note that testing $\delta = (\delta_\alpha, \delta_\beta) = 0$ is the traditional *Behrens-Fisher* problem or the Chow test under heteroscedasticity, when the change-point is known.
7. For the first year (1926) stocks are assigned based on their market values in January 1926, since the market value in December 1925 is not available.

References

Alexander, G.J. and N.L. Chervany, "On the Estimation and Stability of Beta." *Journal of Financial and Quantitative Analysis* 15, 123–137, (1980).

Altman, E.I., B. Jacquillat, and M. Levasseur, "Comparative Analysis of Risk Measures: France and the United States." *Journal of Finance* 29, 1495-1571, (1974).

Baesel, J.B., "On the Assessment of Risk: Some Further Considerations." *Journal of Finance* 29, 1491–4, (1974).

Blume, M.E., "On the Assessement of Risk." *Journal of Finance* 26, 1–10, (1971).

Blume, M.E., "Betas and Their Regression Tendencies." *Journal of Finance* 30, 785-95, (1975).

Bos, T. and Newbold, "An Empirical Investigation of the Possibility of Stochastic Systematic Risk in the Market Model." *Journal of Business* 57, 35–41, (1984).

Brown, P., A.W. Kleidon, and T. Marsh, "New Evidence on the Nature of Size-related Anomalies in Stock Prices." *Journal of Financial Economics* 12, 33–56, (1983).

Chen, S.N., "Beta Nonstationarity, Portfolio Residual Risk and Diversification." *Journal of Financial and Quantitative Analysis* 16, 95–111, (1981).

Fabozzi, F.J. and J.C. Francis, "Beta as a Random Coefficient." *Journal of Financial Quantitative Analysis* 13, 101–116, (1978).

Fabozzi, F.J. and J.C. Francis, "The Effects of Changing Macroeconomic Conditions on the Parameters of the Single Index Market Model." *Journal of Financial and Quantitative Analysis* 14, 351–360, (1979a).

Fabozzi, F.J. and J.C. Francis, "Mutual Fund Risk Statistics for Bull and Bear Markets: An Empirical Examination." *Journal of Finance* 34, 1243–1250, (1979b).

Fama, E. and J.D. MacBeth, "Risk, Return and Equilibrium: Empirical Tests." *Journal of Political Economy* 71, 607–636, (1973).

Gonedes, N.J., "Evidence on the Information Content of Accounting Numbers: Accounting-based and Market-based Estimates of Systematic Risk." *Journal of Financial and Quantitative Analysis* 9, 407–43, (1973).

Ibbotson, R.C., *Stocks, Bonds, Bills, and Inflation: 1990 Yearbook*, Chicago, IL: R.G. Ibbotson Associates, (1990).

Kim, D., "A Bayesian Significance Test on the Stationarity of Regression Parameters." *Biometrika* 78, 667–675, (1991).

Kim, D. and S.J. Kon, "Sequential Parameter Nonstationarity in Stock Market Returns." Working paper, The Graduate School of Business Administration, University of Michigan, (1992).

Levy, R.A., "Stationarity of Beta Coefficients." *Financial Analysts Journal* 27, 52–62, (1971).

Ohlson, J. and B. Rosenberg, "Systematic Risk of the CRSP Equal-weighted Common Stock Index: A History Estimated by Stochastic Parameter Regression." *Journal of Business* 55, 121–145, (1982).

Roenfelt, R.L., G.L. Griepentrog and C.C. Pflaum, "Further Evidence on the Stationarity of Beta Coefficients." *Journal of Financial and Quantitative Analysis* 13, 117–121, (1978).

Roll, R., "A Possible Explanation of the Small Firm Effect." *Journal of Finance* 36, 879–888, (1981).

Sharpe, W.F., "Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk." *Journal of Finance* 19, 425–442, (1964).

Sunder, S., "Stationarity of Market Risk: Random Coefficients Test for Individual Stocks." *Journal of Finance* 35, 883–896, (1980).

Treynor, J.L. and K.K. Mazuy, "Can Mutual Funds Outguess the Market?" *Harvard Business Review* (July–August 1966).